SOLUTIONS OF EXERCISES.

90

SHOW that if x be acute, one value of $\log(1 + \cos 2x + i \sin 2x)$ is $ix + \log(2 \cos x)$.

SOLUTION.

Use the identity

$$\log(A + iB) = \tfrac{1}{2} \log(A^2 + B^2) + i \tan^{-1} \frac{B}{A},$$

$$\therefore \log(1 + \cos 2x + i \sin 2x) = \tfrac{1}{2} \log\{(1 + \cos 2x)^2 + (\sin 2x)^2\}$$

$$+ i \tan^{-1} \frac{\sin 2x}{1 + \cos 2x}$$

$$= \log(2 \cos x) + ix.$$

If $x < \frac{1}{2}\pi$, then $\log(2 \cos a)$ is real; but if $x > \frac{1}{2}\pi$, then $\log(2 \cos x)$ may be replaced by $\log[2 \sin(x - \frac{1}{2}\pi)] + i(2n + 1)\pi$. [*Jesse Pawling, Jr.*]

242

THE curve $\tan x + \tan y = a$ is symmetrical with regard to certain lines parallel to $x + y = 0$.

SOLUTION.

Turning the axes through 45°, the equation becomes

$$\tan \frac{x - y}{\sqrt{2}} + \tan \frac{x + y}{\sqrt{2}} = a,$$

i. e. $$2\left[1 + \tan^2 \frac{y}{\sqrt{2}}\right] \tan \frac{x}{\sqrt{2}} = a\left[1 - \tan^2 \frac{x}{\sqrt{2}} \tan^2 \frac{y}{\sqrt{2}}\right];$$

and the curve is now to be proved symmetrical with regard to certain lines $x = h$. To transfer the origin to $(h, 0)$, put $x + h$ for x; and write for brevity $\tan \frac{x}{\sqrt{2}} = \alpha$, $\tan \frac{y}{\sqrt{2}} = \beta$, $\tan \frac{h}{\sqrt{2}} = \gamma$; then the equation becomes

$$2(1 + \beta^2)(\alpha + \gamma)(1 - \alpha\gamma) = a(1 - \alpha\gamma)^2 - a\beta^2(\alpha + \gamma)^2.$$

The axis of y is an axis of symmetry if this equation be free from odd powers of α, i. e. if

$$2(1 + \beta^2)(1 - \gamma^2) = -2a(\gamma + \beta^2\gamma);$$

hence h is to be found from the equation

$$\frac{2\gamma}{1-\gamma^2} = -\frac{2}{a},$$

giving

$$\tan 2\left[\frac{h}{\sqrt{2}}\right] = -\frac{2}{a}, \quad \text{or} \quad -h\sqrt{2} = \cot^{-1}\tfrac{1}{2}a\,.$$

It follows that the given curve is symmetrical with regard to any of the lines represented by the equation $x + y + \cot^{-1}\frac{a}{2} = 0$, referred to the original axes. These lines are equivalent at intervals of $\frac{\pi}{\sqrt{2}}$ units of length.

[*C. D. Child.*]

294

FIND the equation, on the developed surface of a cone, of the section line made by the intersection of a plane with the cone.

SOLUTION.

Let the equations of the cone and plane be

$$x^2 + y^2 = z^2 \tan^2\alpha\,, \tag{1}$$

$$x\cos\beta + z\sin\beta = p\,, \tag{2}$$

and let φ be the azimuth of the generator passing through (x, y, z), i. e. the angle which the plane of the generator and the axis of z, makes with the plane (xz); then when the cone is developed, the angle between this generator and the generator of zero azimuth, is easily seen to be $\varphi \sin\alpha$; and the polar equation of the section line may be obtained by eliminating x, y, z, φ, from (1), (2) above, and (3), (4), (5) following:

$$\theta = \varphi\sin\alpha\,, \tag{3}$$

$$y = x\tan\varphi\,, \tag{4}$$

$$r^2 = x^2 + y^2 + z^2\,, \tag{5}$$

giving for the required equation

$$r\,[\tan\beta + \tan\alpha\cos(\theta\,\mathrm{cosec}\,\alpha)] = p\sec\alpha\sec\beta\,.$$

The form of the curve depends on the relative values of α and β; for instance, if $\tan\beta$ is numerically less than $\tan\alpha$, there are two values of θ (between the extreme limits 0 and $2\pi\sin\alpha$ for the developed surface) that give infinite values to r.

If the curve be extended beyond these limits, it will repeat itself, since r is a periodic function of θ, having the period $2\pi \sin \alpha$; and the waves will overlap unless cosec α is an integer. In the latter case the curve is algebraic, and of the degree 2 cosec α; the numbers of maxima and minima of the radii vectores being each equal to cosec α, when the value of β is such that the curve is closed. When cosec α is m/n, an improper fraction in its lowest terms, the degree of the curve is $2m$. [*James McMahon.*]

355

Required the locus of the point in the normal to a conic, which is equally distant from the focus and the foot of the normal. [*Geo. R. Dean.*]

CORRECTED SOLUTION.

Taking the origin at the center of the ellipse we have for the equation of the normal

$$y - b \sin \varphi - \frac{a}{b} \tan \varphi \, (x - a \cos \varphi) \, ,$$

or

$$y = \frac{a}{b} \tan \varphi - \frac{a^2 - b^2}{b} \sin \varphi .$$

The locus required is the intersection of this normal with the line

$$y = - \frac{a}{b} \tan \varphi \, (x - ae) \, .$$

Eliminating φ, we get

$$y = 0 \, ,$$

and

$$\{(x - ae + 1)^2 - a^2e^4\} \, (x - ae)^2 + (1 - e^2) \, y^2 \, (x - ae + 1)^2 = 0 \, ,$$

the x-axis, which is excluded, and a curve having $(x - ae + 1)^2 = 0$ for the equation of two coincident asymptotes. The curve cuts the x-axis at the points $x = ae \, (1 \pm e) - 1$, and the focus is a conjugate point. [*Geo. R. Dean.*]

359

Four normals can be drawn from a point to a limaçon; if the feet of two of the normals lie on a line through the node, the feet of the other two lie on a line through the focus. [*Frank Morley.*]

SOLUTION.

The limaçon, regarded as traced by a point attached to a circle which rolls on an equal circle, is expressed at once by the equation

$$x = 2at - \beta t^2 \, ,$$

where α, β are real, and $|t| = 1$. Writing y for the conjugate of x, the normal is

$$x(\alpha t - \beta) - yt^3(\alpha - \beta t) = \alpha\beta(t^3 - t).$$

Let the roots of this equation in t be t_1, t_2, t_3, t_4; then the conditions that the normals at these points meet are

$$\Sigma t_1 t_2 = 0, \quad \Sigma t_1 + \Sigma t_1 t_2 t_3 = (1 + t_1 t_2 t_3 t_4)\,\alpha/\beta.$$

If t_3, t_4 are ends of a nodal chord, then $t_3 = -t_4$, and therefore

$$t_1 t_2 = -t_3 t_4,$$

and

$$(t_1 + t_2)(1 + t_3 t_4) = (1 - t_3^2 t_4^2)\,\alpha/\beta,$$

so that

$$t_1 + t_2 = (1 + t_1 t_2)\,\alpha/\beta.$$

Hence

$$(\alpha - \beta t_1)/(\alpha - \beta/t_1) = -t_1 t_2 = (\alpha - \beta t_2)/(\alpha - \beta/t_2).$$

Hence the amplitudes of $(\alpha - \beta t_1)^2$ and $(\alpha - \beta t_2)^2$ are equal. Now the focus is α^2/β; and the stroke from the focus to any point t is $-(\alpha - \beta t)^2/\beta$; and the statement is proved. [*Frank Morley.*]

360

NORMALS at the ends of a nodal chord of a given limaçon mark off an involution on the axis of the curve. [*Frank Morley.*]

SOLUTION.

For the notation see solution of Ex. 359. The normal at t_1 meets the real axis at ν_1, where

$$\nu_1\{\alpha t_1 - \beta(1 + t_1^2)\} = -\alpha\beta t_1,$$

or

$$1/\nu_1 + 1/\beta = (t_1 + t_1^{-1})/\alpha.$$

Therefore, when $t_1 + t_2 = 0$,

$$1/\nu_1 + 1/\nu_2 = -2/\beta.$$

That is, the points are harmonic with the fixed points 0 and $-\beta$.

[*Frank Morley.*]

361

LET r be the base of a system of numeration. Find the condition that in the quotient of the number

$$A = aaa \ldots a \qquad (r - 1 \text{ places})$$

divided by $r - 1$, there shall appear all but one of the digits of the system (0 excluded), and determine the lacking digit. [*Edgar H. Johnson.*]

SOLUTION.

If 0 appear in this quotient the division must be exact before the end is reached. In order that any number be exactly divisible by $r-1$, it is necessary and sufficient that the sum of its digits be so divisible; so that if 0 appears, a and $r-1$ must have a common divisor, and conversely. By application of the same principle it is easily seen that this is also the necessary and sufficient condition that in this quotient two of the digits be the same. So that if, and only if, a and $r-1$ are relative primes, none of the $r-2$ digits in the quotient will be the same and one digit besides 0 will be lacking.

If $r-1$ is prime every digit, of course, satisfies the required condition,

$$r^{r-2} + r^{r-3} + \dots r + 1/r - 1 = r^{r-3} + 2\,r^{r-4} + 3\,r^{r-5} + \dots (r-3)\,r + (r-1)\,;$$

So that the quotient of the number

$$111 \dots 1 \qquad [r-1 \text{ places}]$$

divided by $r-1$ is the number

$$N = 123 \dots r-4\ r-3\ r-1\,.$$

If from the dividend we subtract the quotient N, the remainder

$$r-1\ r-2\ r-3 \dots 432$$

is the quotient of the number

$$r-2\ r-2\ r-2 \dots r-2 \qquad [r-1 \text{ places}]$$

divided by $r-1$.

To find the quotient of A divided by $r-1$ we have only to multiply N by a.

The sum of all the digits in the system is $\frac{r(r-1)}{2}$. The sum of the digits in N is $\frac{r^2-3r+4}{2}$.

The remainder after the division of any number by $r-1$ is the same as after division of the sum of its digits by $r-1$. Applying this to N we see that the remainder is 1 if r is even and $\frac{r+1}{2}$ if r is odd.

When r is even the sum of the digits in aN is congruent to a and the sum of all the digits in the system is congruent to 0. Hence the lacking digit in aN is $r-1-a$.

When r is odd the sum of the digits in aN is congruent to $\frac{a(r+1)}{2}$; and the sum of all the digits in the system is congruent to $\frac{r-1}{2}$. The miss-

ing digit is, as before, $r - 1 - a$; for

$$r - 1 - a + \frac{a(r+1)}{2} \equiv \frac{a(r-1)}{2} \equiv \frac{r-1}{2} \pmod{r-1},$$

it being noticed that $\frac{(a-1)(r-1)}{2} \equiv 0 \pmod{r-1}$, because a is odd, being relative prime to $r - 1$. [*Edgar H. Johnson.*]

364

FIND two complete integrals of the equation

$$\left[\frac{\partial z}{\partial x}\right]^2 + \left[\frac{\partial z}{\partial y}\right]^2 = \frac{x-y}{z}.$$ [*Geo. R. Dean.*]

SOLUTION.

Writing the equation in the form $z(p^2 + q^2) - (x - y) = 0$, and using Charpit's method, we have the subsidiary equations,

$$\frac{dx}{2zp} = \frac{dy}{2zq} = \frac{dz}{2z(p^2+q^2)} = \frac{-dp}{-1+p(p^2+q^2)} = \frac{-dq}{1+q(p^2+q^2)}.$$

1. From the relations between dz, dp and dq, we get

$$\frac{dz}{2z} = \frac{-d(p+q)}{p+q};$$

whence

$$(p+q)^2 = \frac{c}{z}.$$

Combining this equation with the original, and substituting the values of p and q in $dz = pdx + qdy$, we have

$$2z^{\frac{1}{2}}dz = [c^{\frac{1}{2}} + \{2(x-y)+c\}^{\frac{1}{2}}]\,dx + [c^{\frac{1}{2}} - \{2(x-y)+c\}^{\frac{1}{2}}]\,dy,$$

the integral of which is

$$\tfrac{4}{3}z^{\frac{3}{2}} = c^{\frac{1}{2}}(x+y) + \tfrac{1}{3}\{2(x-y)-c\}^{\frac{3}{2}}.$$

2. From the relations between dx, dz, and dp we get

$$p = \left[\frac{x-c}{z}\right]^{\frac{1}{2}}, \quad q = \left[\frac{c-y}{z}\right]^{\frac{1}{2}},$$

which substituted in $dz = pdx + qdy$, give after integration

$$z^{\frac{3}{2}} = (x-c)^{\frac{3}{2}} - (c-y)^{\frac{3}{2}} + c.$$ [*Geo. R. Dean.*]

365

SHOW that

$$x^3 + y^3 + z^3 - 3xyz = a^3$$

is a surface of revolution, and find its axis. [*Geo. R. Dean.*]

SOLUTION.

The differential equation of a surface of revolution is

$$(ly - mx) + (ny - mz)p + (lz - mx)q = 0\,,$$

where l, m, and n are the direction-cosines of the axis, and

$$p = \frac{\partial z}{\partial x}\,, \quad q = \frac{\partial x}{\partial y}\,.$$

The values of p and q for the given equation give, when substituted in the differential equation,

$$(ly - mx)(z^2 - xy) + (ny - mz)(x^2 - yz) + (lz - nx)(y^2 - xz) = 0\,,$$

which is satisfied when $l = m = n$. Hence the equations of the axis are

$$x = y = z\,.$$

[*Geo. R. Dean.*]

366

SHOW that for any parabola $y = x^2 + ax + b$, the area included between the curve, any two ordinates, and the x-axis, is equal to the product of the ordinate midway between the bounding ordinates and the interval between them, plus one-twelfth the cube of this interval. [*W. H. Echols.*]

SOLUTION.

While many simple solutions may be given, the following is not without interest as an application of the general formula (23), Annals of Mathematics, Vol. VII, p. 21.*

Let

$$f(x) = x^2 + ax + b\,.$$

We have

$$\int_q^p f(x)\,dx = (p - q)f(\alpha) + Af'(\beta) + Bf''(\gamma)\,,$$

wherein α, β, γ are arbitrary, and

$$A = \frac{1}{2!}(p^2 - q^2) - \alpha(p - q)\,,$$

$$B = \frac{1}{3!}(p^3 - q^3) + \frac{1}{2!}\beta(p^2 - q^2) - \alpha\beta(p - q) + \frac{1}{2!}\alpha^2(p - q)\,.$$

Make $\alpha = \frac{1}{2}(p + q)$, then $A = 0$ and $B = \frac{1}{24}(p - q)^3$, while $f''(\gamma) = 2$.

* Otherwise, immediately by inspection of formulæ (45), p. 33; (63), (64), p. 36; (68), p. 37.

Whence

$$\int_q^p f(x)dx = (p-q)f\left[\frac{p+q}{2}\right] + \tfrac{1}{12}(p-q)^3.$$

[*W. H. Echols.*]

Also solved by Messrs. Geo. R. Dean, Eric Doolittle, and H. Y. Benedict.

EXERCISES.

369

SOLVE the equation

$$\frac{d^2y}{dx^2} - y = e^{\frac{1}{2}x^2}.$$

[*Geo. R. Dean.*]

370

FROM a point A on the equator a northeast rhumb line is drawn; find at what latitude it again strikes the meridian of A, and express the length of the rhumb line in radii. [*James McMahon.*]

371

PROVE that the ratio of

$$\begin{vmatrix} \sigma_1(2u_1), & \sigma_1(2u_2), & \sigma_1(2u_3) \\ \sigma_2(2u_1), & \sigma_2(2u_2), & \sigma_2(2u_3) \\ \sigma_3(2u_1), & \sigma_3(2u_2), & \sigma_3(2u_3) \end{vmatrix}$$

to

$$\sigma(u_2+u_3)\,\sigma(u_3+u_1)\,\sigma(u_1+u_2)\,\sigma(u_2-u_3)\,\sigma(u_3-u_1)\,\sigma(u_1-u_2)$$

is independent of the arguments u_λ; and that its value is

$$4(e_2-e_3)(e_3-e_1)(e_1-e_2);$$

the notation being that of Weierstrass. [*Frank Morley.*]

372

WHEN the bilinear invariant of two binary n-ics is zero, we say that the n-ics are *apolar*. When also the n-ics coincide we say that either is self-apolar. And we may apply the same adjectives to the sets of n points (or n-*ads*) which represent the zeros of the n-ics. Any odd set of points is, we know, self-apolar (Salmon's Higher Algebra, § 153). Prove that an even set is self-apolar when the first polar of any point of the set, with regard to the rest, is self-apolar. [*Frank Morley.*]